

Extremal eigenvalues of the Laplacian in a conformal class of metrics : the "conformal spectrum"

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Abstract

Let M be a compact connected manifold of dimension n endowed with a conformal class C of Riemannian metrics of volume one. For any integer $k \geq 0$, we consider the conformal invariant $\lambda_k^c(C)$ defined as the supremum of the k -th eigenvalue $\lambda_k(g)$ of the Laplace-Beltrami operator Δ_g , where g runs over C .

First, we give a sharp universal lower bound for $\lambda_k^c(C)$ extending to all k a result obtained by Friedlander and Nadirashvili for $k = 1$. Then, we show that the sequence $\{\lambda_k^c(C)\}$, that we call "conformal spectrum", is strictly increasing and satisfies, $\forall k \geq 0$, $\lambda_{k+1}^c(C)^{n/2} - \lambda_k^c(C)^{n/2} \geq n^{n/2}\omega_n$, where ω_n is the volume of the n -dimensional standard sphere.

When M is an orientable surface of genus γ , we also consider the supremum $\lambda_k^{top}(\gamma)$ of $\lambda_k(g)$ over the set of all the area one Riemannian metrics on M , and study the behavior of $\lambda_k^{top}(\gamma)$ in terms of γ .

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1 Introduction and statement of results

Let M be a closed connected differentiable manifold of dimension $n \geq 2$. Given a Riemannian metric g on M , let

$$\text{spec}(g) = \{0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \dots \leq \lambda_k(g) \leq \dots\}$$

be the spectrum of the Laplace-Beltrami operator defined by g .

One of the main topics in spectral geometry is the study of the variational properties of the functional $g \mapsto \lambda_k(g)$ and the finding out of extremal geometries for λ_k . Problems of this kind were first studied in the setting of Euclidean domains where many Faber-Krahn type inequalities have been established (see [He] for a recent survey). In the case of closed manifolds which interests us here, the first result was obtained by Hersch [H]: on the 2-sphere \mathbb{S}^2 , the standard metric g_s maximizes λ_1 among all the Riemannian metrics of the same area. Moreover, g_s is, up to isometry, the unique maximizer.

Recall that the behavior of λ_k under scaling of the metric is given by $\lambda_k(cg) = \lambda_k(g)/c$. Hence, a normalization is required. The metric invariant usually considered for this normalization is the volume $V(g)$. Therefore, we denote by $\mathcal{M}(M)$ the set of all Riemannian metrics of *volume one* on M and, for any $g \in \mathcal{M}(M)$, we set

$$[g] = \{g' \in \mathcal{M}(M) \mid g' \text{ is conformal to } g\}.$$

It is well known that if M is of dimension $n \geq 3$, then, $\forall k \geq 1$, λ_k is not bounded on $\mathcal{M}(M)$ (see [CD]). On the other hand, Korevaar [K] showed that in dimension 2, λ_k is bounded on $\mathcal{M}(M)$ and that, in all dimensions, the restriction of λ_k to any conformal class of metrics of fixed volume is bounded. Hence, for any natural integer k and any conformal class of metrics $[g]$ on M , we define the *conformal k -th eigenvalue* of $(M, [g])$ to be

$$\lambda_k^c(M, [g]) = \sup_{g' \in [g]} \lambda_k(g') = \sup \{ \lambda_k(g') V(g')^{2/n} \mid g' \text{ is conformal to } g \}.$$

The sequence $\{\lambda_k^c(M, [g])\}$ constitutes the *conformal spectrum* of $(M, [g])$.

In dimension 2, one can also define a *topological spectrum* by setting, for any genus γ and any integer $k \geq 0$,

$$\lambda_k^{\text{top}}(\gamma) = \sup \{ \lambda_k(g) \mid g \in \mathcal{M}(M_\gamma) \},$$

M_γ being an orientable compact surface of genus γ .

The aim of this paper is to emphasize some properties of the conformal and topological spectra. Let us first recall some of the known results. Actually, most of them concern only the first positive eigenvalue. Indeed, the

result of Hersch mentioned above reads: $\lambda_1^{top}(0) = \lambda_1(g_s) = 8\pi$, where g_s is the standard metric normalized to volume one. For genus one surfaces, Nadirashvili [N1] showed that $\lambda_1^{top}(1) = \lambda_1(g_e) = 8\pi^2/\sqrt{3}$, where g_e is the flat metric induced on the 2-torus from an equilateral lattice of \mathbb{R}^2 . For arbitrary genus, Yang and Yau [YY] proved the following inequality (see also [EI1]):

$$\lambda_1^{top}(\gamma) \leq 8\pi \left[\frac{\gamma + 3}{2} \right],$$

where $[\]$ denotes the integer part, and Korevaar [K] showed the existence of a universal constant C such that, $\forall k \geq 0$,

$$\lambda_k^{top}(\gamma) \leq C(\gamma + 1)k.$$

In higher dimension, Korevaar also obtained in [K] the estimate :

$$\lambda_k^c(M, [g]) \leq C([g])k^{2/n}$$

for some constant $C([g])$ depending on n and on a lower bound of $Ric\ d^2$, where Ric is the Ricci curvature and d is the diameter of g or of another representative of $[g]$.

Regarding the conformal first eigenvalue, the second author and Ilias [EI2] gave a sufficient condition for a Riemannian metric g to maximize λ_1 in its conformal class $[g]$: if there exists a family f_1, f_2, \dots, f_p of first eigenfunctions satisfying $\sum_i df_i \otimes df_i = g$, then $\lambda_1^c(M, [g]) = \lambda_1(g)$. This condition is fulfilled in particular by the metric of any homogeneous Riemannian space with irreducible isotropy representation. For instance, the first conformal eigenvalues of the rank one symmetric spaces endowed with their standard conformal classes $[g_s]$, are given by

- $\lambda_1^c(\mathbb{S}^n, [g_s]) = n\omega_n^{2/n}$, where ω_n is the volume of the n -dimensional Euclidean sphere of radius one,
- $\lambda_1^c(\mathbb{R}P^n, [g_s]) = 2^{\frac{n-2}{n}}(n+1)\omega_n^{2/n}$,
- $\lambda_1^c(\mathbb{C}P^d, [g_s]) = 4\pi(d+1)d!^{-1/d}$,
- $\lambda_1^c(\mathbb{H}P^d, [g_s]) = 8\pi(d+1)(2d+1)!^{-1/2d}$,
- $\lambda_1^c(\mathbb{C}aP^2, [g_s]) = 48\pi(\frac{6}{11})^{1/8} = 8\pi\sqrt{6}(\frac{9}{385})^{1/8}$.

On the other hand, Ilias, Ros and the second author [EIR] proved that if $\Gamma = \mathbb{Z}e_1 + \mathbb{Z}e_2 \subset \mathbb{R}^2$ is a lattice such that $|e_1| = |e_2|$, then the corresponding flat metric g_Γ on \mathbb{T}^2 satisfies $\lambda_1^c(\mathbb{T}^2, [g_\Gamma]) = \lambda_1(g_\Gamma)$. A higher dimensional

version of this result was also established in [EI3]. Nevertheless, the authors [CE] have showed that when the length ratio $|e_2|/|e_1|$ of the vectors e_1 and e_2 is sufficiently far from 1, then $\lambda_1^c(\mathbb{T}^2, [g_r]) > \lambda_1(g_r)$, that is, g_r does not maximize λ_1 on $[g_r]$.

Finally, the following relationship between $\lambda_1^c(M, [g])$ and the conformal volume $V_c(M, [g])$ is due to Li and Yau [LY] in dimension 2, and to the second author and Ilias [EI2] in all dimensions:

$$\lambda_1^c(M, [g]) \leq nV_c(M, [g])^{2/n}.$$

Our first result states that among all the possible conformal classes of metrics on manifolds, the standard conformal class of the sphere is the one having the lowest conformal spectrum.

Theorem A *For any conformal class $[g]$ on M and any integer $k \geq 0$,*

$$\lambda_k^c(M, [g]) \geq \lambda_k^c(\mathbb{S}^n, [g_s]).$$

Although the eigenvalues of a given Riemannian metric may have non-trivial multiplicities, the conformal eigenvalues are all simple: the conformal spectrum consists on a strictly increasing sequence, and, moreover, the gap between two consecutive conformal eigenvalues is uniformly bounded. Precisely, we have the following theorem:

Theorem B *For any conformal class $[g]$ on M and any integer $k \geq 0$,*

$$\lambda_{k+1}^c(M, [g])^{n/2} - \lambda_k^c(M, [g])^{n/2} \geq \lambda_1^c(\mathbb{S}^n, [g_s]) = n^{n/2}\omega_n,$$

where ω_n is the volume of the n -dimensional Euclidean sphere of radius one.

An immediate consequence of these two theorems is the following explicit estimate of $\lambda_k^c(M, [g])$:

Corollary 1 *For any conformal class $[g]$ on M and any integer $k \geq 0$,*

$$\lambda_k^c(M, [g]) \geq n\omega_n^{2/n}k^{2/n}.$$

Note that, for $k = 1$, this last inequality has been recently proved by Friedlander and Nadirashvili [FN] (see also [CE]). However, our method is more general and simpler.

Of course, in the particular case of the n -sphere \mathbb{S}^n endowed with its standard conformal class $[g_s]$, the equality holds in this inequality for $k = 1$. The equality also holds for $k = 2$ on \mathbb{S}^2 as it was recently proven by Nadirashvili [N2], i.e. $\lambda_2^c(\mathbb{S}^2, [g_s]) = 8\pi$.

Combined with the Korevaar estimate quoted above, Corollary 1 gives

$$n\omega_n^{2/n}k^{2/n} \leq \lambda_k^c(M, [g]) \leq C([g])k^{2/n}.$$

Corollary 1 implies also that, if the k -th eigenvalue $\lambda_k(g)$ of a metric g is less than $n\omega_n^{2/n}k^{2/n}$, then g does not maximize λ_k on its conformal class $[g]$. In particular, we have the following (negative) answer to a question of Yau concerning \mathbb{S}^2 (see [Y], p. 686):

Corollary 2 *For any integer $k \geq 2$, the standard metric g_s of \mathbb{S}^2 does not maximize λ_k , that is there exists a metric g_k of volume one on \mathbb{S}^2 such that*

$$\lambda_k(g_k) > \lambda_k(g_s).$$

Indeed, $\lambda_k(g_s) = 4\pi[\sqrt{k}](\lceil\sqrt{k}\rceil + 1)$, where $\lceil\sqrt{k}\rceil$ is the integer part of \sqrt{k} , while $\lambda_k^c(\mathbb{S}^2, [g_s]) \geq 8\pi k$. The same calculations show that, on \mathbb{S}^3 , for any $k \geq 2$, we have $\lambda_k(g_s) < 3\omega_3^{2/3}k^{2/3}$, and then *the k -th eigenvalue does not achieve its maximum on $[g_s]$ at g_s .*

On the other hand, in any dimension we have, $\lambda_1^c(\mathbb{S}^n, [g_s]) = \lambda_1(g_s) = \lambda_2(g_s) = \dots = \lambda_{n+1}(g_s)$. Consequently, $\forall k \in [2, n+1]$, *the standard metric g_s of \mathbb{S}^n does not maximize λ_k in its conformal class.*

In [EI3] (see also [EI4] and [N]), Ilias and the second author studied the property for a Riemannian metric to be critical (in a generalized sense) for the functional $g \mapsto \lambda_k(g)$. A consequence of their results is that, if a metric g is extremal for λ_k under conformal deformations, then the multiplicity of $\lambda_k(g)$ is at least 2, which means that $\lambda_k(g) = \lambda_{k+1}(g)$ or $\lambda_k(g) = \lambda_{k-1}(g)$. Combined with Theorem B, this fact yields:

Corollary 3 *If a Riemannian metric g maximizes λ_1 on its conformal class $[g]$, then it does not maximize λ_2 on $[g]$. More generally, a Riemannian metric g cannot maximize simultaneously three consecutive eigenvalues λ_k , λ_{k+1} and λ_{k+2} on $[g]$.*

Applying Theorem B to an orientable surface M_γ of genus γ , we obtain the following result concerning the topological spectrum.

Corollary 4 *For any fixed genus γ and any integer $k \geq 0$,*

$$\lambda_{k+1}^{top}(\gamma) - \lambda_k^{top}(\gamma) \geq 8\pi,$$

and

$$\lambda_k^{top}(\gamma) \geq 8(k-1)\pi + \lambda_1^{top}(\gamma) \geq 8k\pi.$$

Our last result answers the following question: how does $\lambda_k^{top}(\gamma)$ behave as γ increases?

Theorem C *For any fixed integer $k \geq 0$, the function $\gamma \mapsto \lambda_k^{top}(\gamma)$ is increasing, that is*

$$\lambda_k^{top}(\gamma + 1) \geq \lambda_k^{top}(\gamma).$$

Recently, Brooks and Makover [BM] proved that, if C is the Selberg constant, then, for any $\varepsilon > 0$, there exists an integer N such that any compact orientable surface of genus $\gamma \geq N$ admits a hyperbolic metric g with $\lambda_1(g) \geq C - \varepsilon$. As the area of such a hyperbolic surface is equal to $4\pi(\gamma - 1)$, it follows that $\lambda_1^{top}(\gamma) \geq 4(C - \varepsilon)\pi(\gamma - 1)$. Although the Selberg conjecture " $C = 1/4$ " is still open, it has been proved that $C \geq 171/784 > 1/5$ ([LRS]). Hence, for sufficiently large γ , $\lambda_1^{top}(\gamma) \geq \frac{4}{5}\pi(\gamma - 1)$ and then, $\forall k \geq 0$,

$$\lambda_k^{top}(\gamma) \geq \frac{4}{5}\pi(\gamma - 1) + 8\pi(k - 1).$$

2 Preliminary results

Roughly speaking, the proof of the two theorems A and B lies on the following idea, also used in [FN]: locally, a Riemannian manifold (M, g) is almost Euclidean, and, consequently, almost conformal to the sphere endowed with its standard metric g_s . Then, given a metric h in the conformal class of the standard metric of the sphere, it will be possible to construct a conformal deformation of (M, g) around a point to make this neighborhood arbitrarily close (in some sense) to (\mathbb{S}^n, h) . To make these points precise, we will establish some preliminary results and recall some facts from literature.

Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds of the same dimension $n \geq 2$. Let us suppose that there exists, for each $i \leq 2$, a point $x_i \in M_i$ such that the metric g_i is flat in a neighborhood of x_i . Therefore, for sufficiently small $\varepsilon > 0$, the geodesic balls $B_1(x_1, \varepsilon) \subset M_1$ and $B_2(x_2, \varepsilon) \subset M_2$ are both isometric to a Euclidean ball. If $\Phi_\varepsilon : \partial B_1(x_1, \varepsilon) \rightarrow \partial B_2(x_2, \varepsilon)$ is an induced isometry between their boundaries, then we obtain a new closed manifold M_ε by glueing $M_1 \setminus B_1(x_1, \varepsilon)$ to $M_2 \setminus B_2(x_2, \varepsilon)$ along Φ_ε . Let $\{\lambda_k(\varepsilon); k \geq 0\}$ be the spectrum of the natural Laplacian Δ_ε of M_ε (see [A]) associated to the piecewise smooth metric g_ε which coincide with g_i on $M_i \setminus B_i(x_i, \varepsilon)$, and let $\{\Lambda_k; k \geq 0\}$ be the reordered union of the spectra of (M_1, g_1) and (M_2, g_2) . Then we have the following:

Lemma 2.1 *For all $k \in \mathbb{N}$, we have*

$$\lim_{\varepsilon \rightarrow 0} \lambda_k(\varepsilon) = \Lambda_k.$$

This Lemma is a direct consequence of the min-max principle and the results of [A]. Indeed, let $\{\mu_k(\varepsilon); k \geq 0\}$ (resp. $\{\nu_k(\varepsilon); k \geq 0\}$) be the reordered union of the spectra of $M_1 \setminus B_1(x_1, \varepsilon)$ and $M_2 \setminus B_2(x_2, \varepsilon)$ with the Dirichlet (resp. Neumann) boundary condition. The following inequalities are direct consequences of the min-max principle:

$$\nu_k(\varepsilon) \leq \lambda_k(\varepsilon) \leq \mu_k(\varepsilon).$$

On the other hand, for each $i \leq 2$, the spectrum of $M_i \setminus B_i(x_i, \varepsilon)$ with the Dirichlet or the Neumann boundary condition, converges, as $\varepsilon \rightarrow 0$, to the spectrum of the closed manifold (M_i, g_i) (see [A]).

We also have a similar result in the case where we deform the metric on M_2 so that it collapses to a point. Indeed, by changing the scale of the metric g_2 if necessary, we may assume that the radius one geodesic ball $B_2(x_2, 1)$ of (M_2, g_2) is contained in the flat neighborhood of x_2 . Now, if we replace on M_2 the metric g_2 by $g_2(\varepsilon) = \varepsilon^2 g_2$, then the geodesic ball $B_1(x_1, \varepsilon)$ of (M_1, g_1) becomes isometric to the ball $B_2(x_2, 1)$ of (M_2, g_2) endowed with the new metric $g_2(\varepsilon)$. Again, we consider the manifold M_ε obtained by glueing $M_1 \setminus B_1(x_1, \varepsilon)$ to $M_2 \setminus B_2(x_2, 1)$ and endow it with the metric g_ε which coincide with g_1 on $M_1 \setminus B_1(x_1, \varepsilon)$ and with $g_2(\varepsilon)$ on $M_2 \setminus B_2(x_2, 1)$. When ε goes to zero, Takahashi [T] proved that the spectrum of $(M_\varepsilon, g_\varepsilon)$ converges to the spectrum of (M_1, g_1) .

Lemma 2.2 ([T]) *For all $k \in \mathbb{N}$, we have*

$$\lim_{\varepsilon \rightarrow 0} \lambda_k(M_\varepsilon, g_\varepsilon) = \lambda_k(M_1, g_1).$$

Notice that, in dimension $n \geq 3$, this result can also be derived from a theorem of Colin de Verdière [CV].

The proof of the theorems will also use the fact that the spectrum of a Riemannian metric does not change much when we replace this latter by a quasi-isometric one with a quasi-isometry ratio close to one. Recall that two Riemannian metrics g_1 and g_2 on a compact manifold M are said to be α -quasi-isometric, where $\alpha \geq 1$, if, for any tangent vector $v \in TM$, $v \neq 0$,

$$\frac{1}{\alpha^2} \leq \frac{g_1(v, v)}{g_2(v, v)} \leq \alpha^2.$$

The spectra of g_1 and g_2 are then related by the following inequalities (see [D]): $\forall k \in \mathbb{N}^*$,

$$\frac{1}{\alpha^{2(n+1)}} \leq \frac{\lambda_k(g_1)}{\lambda_k(g_2)} \leq \alpha^{2(n+1)},$$

while their volumes satisfy

$$\frac{1}{\alpha^n} \leq \frac{V(g_1)}{V(g_2)} \leq \alpha^n.$$

The two following immediate observations will be useful in the sequel:

- O1 If g_1 and g_2 are α -quasi-isometric, then, for any positive smooth function f on M , the conformal metrics $f^2 g_1$ and $f^2 g_2$ are also α -quasi-isometric.
- O2 If f_1 and f_2 are two positive functions on M such that $\alpha^{-1} \leq \frac{f_1(x)}{f_2(x)} \leq \alpha$, then, for any metric g on M , the metrics $f_1^2 g$ and $f_2^2 g$ are α -quasi-isometric.

Lemma 2.3 *Let (M, g) be a compact Riemannian manifold and let x_0 be a point of M .*

(i) For any positive δ , there exists a Riemannian metric g_δ which is flat in a neighborhood of x_0 and $(1 + \delta)$ -quasi-isometric to g on M .

(ii) If, in addition, g is conformally flat in a neighborhood of x_0 , then the metric g_δ can also be chosen to be conformal to g .

Proof: (i) In a normal coordinates system centered at x_0 , we have

$$g_{ij}(x) = \delta_{ij} + O(|x|^2).$$

Hence, it is clear that one can construct an adequate g_δ by choosing it equal to δ_{ij} in a geodesic ball $B(x_0, r_\delta)$ of sufficiently small radius r_δ , and equal to g in $M \setminus B(x_0, 2r_\delta)$.

(ii) In the case where g is conformally flat in a neighborhood of x_0 , we have the local expression

$$g_{ij} = f^2(x) \delta_{ij},$$

where f is a smooth function defined in a neighbourhood of x_0 and such that $f(x_0) = 1$. Thus, it suffices to take $g_\delta = \varphi_\delta g$, where φ_δ is a positive smooth function on M such that $\varphi_\delta = f^{-2}$ in a sufficiently small ball $B(x_0, r_\delta)$, and $\varphi_\delta = 1$ in $M \setminus B(x_0, 2r_\delta)$.

3 Proof of the theorems

Let us start with the following elementary construction which will be useful in the sequel.

Construction: Recall that the standard metric g_s of the sphere \mathbb{S}^n is expressed (via the stereographic projection with respect to the north pole) by

$$g(x) = \frac{4}{(1 + \|x\|^2)^2} g_{euc},$$

g_{euc} being the Euclidean metric, while, given a positive number R , the metric

$$g_R(x) = \begin{cases} \frac{4}{(1 + \|x\|^2)^2} g_{euc} & \text{if } \|x\| \leq R \\ \frac{4R^4}{(1 + R^2)^2 \|x\|^4} g_{euc} & \text{if } \|x\| \geq R \end{cases}$$

corresponds to a metric on \mathbb{S}^n which is flat in a ball around the north pole N , and coincide with g_s outside this ball. The radius of this latter depends on R and an easy calculation shows that the set $\{\|x\| \geq R\}$, endowed with g_R , is isometrically equivalent to an Euclidean ball $D_{\varepsilon(R)}$ of radius $\varepsilon(R) = \frac{2R}{(1+R^2)}$.

Now, for a metric h conformal to g_s , we may consider a positive smooth function f , with $f = 1$ on $\{\|x\| \geq R\}$, so that the metric $f^2(x)g_R$ represents h outside the flat ball $D_{\varepsilon(R)}$. Moreover, up to a scaling of the variable x , it is possible to prescribe the radius of the flat ball. Indeed, it suffices to consider, for any positive ρ , the metric

$$g_{f,R,\rho}(x) = f^2\left(\frac{Rx}{\rho}\right)g_R\left(\frac{Rx}{\rho}\right) = \begin{cases} f^2\left(\frac{Rx}{\rho}\right)\frac{R^2}{\rho^2}\frac{4}{(1+\frac{R^2}{\rho^2}\|x\|^2)^2}g_{euc} & \text{if } \|x\| \leq \rho \\ \frac{4R^2\rho^2}{(1+R^2)^2\|x\|^4}g_{euc} & \text{if } \|x\| \geq \rho \end{cases}$$

Note that if $\|x\| = \rho$, then the metric at x is $\frac{\varepsilon^2(R)}{\rho^2}g_{euc}$.

Before going further into the proof, let us note that the conformal eigenvalues are not necessarily achieved by smooth metrics, so that we will always work with smooth Riemannian metrics whose eigenvalues are almost extremal for the problem we study, and then pass to the limit.

Proof of Theorem A: Let (M, g) be a n -dimensional compact Riemannian manifold of volume 1 and let k be a positive integer. Let us fix a positive real number δ .

On the sphere \mathbb{S}^n , consider a Riemannian metric $h \in [g_s]$ of volume one satisfying

$$\lambda_k(h) \geq \lambda_k^c(\mathbb{S}^n, [g_s]) - \delta.$$

As it belongs to the standard conformal class $[g_s]$, the metric h is locally conformally flat and, applying Lemma 2.3 (ii), there exists a metric h_δ , conformal

to g_s , flat in the geodesic ball $D_{r_\delta} \subset \mathbb{S}^n$ of radius r_δ and $(1+\delta)$ -quasi-isometric to h . In particular, h_δ satisfies

$$\begin{aligned}\lambda_k(h_\delta)V(h_\delta)^{2/n} &\geq (1+\delta)^{-2(n+2)}\lambda_k(h)V(h)^{2/n} \\ &\geq (1+\delta)^{-2(n+2)}(\lambda_k^c(\mathbb{S}^n, [g_s]) - \delta).\end{aligned}$$

On the other hand, let g_δ be a metric on M satisfying the conditions of Lemma 2.3 (i). We multiply g_δ by a constant C_δ^2 so that the geodesic ball $B(x_0, 2)$ becomes flat. Now, as explained in the construction above, for any positive $\varepsilon < r_\delta$, there exists on M a metric $f_\varepsilon^2 g_\delta$ conformal to $C_\delta^2 g_\delta$ such that

- The closed ball $\bar{B}(x_0, 1) \subset M$ endowed with $f_\varepsilon^2 g_\delta$ becomes isometric to $(\mathbb{S}^n \setminus D_\varepsilon, h_\delta)$.
- The metric $f_\varepsilon^2 g_\delta$ coincide with $\varepsilon^2 C_\delta^2 g_\delta$ on $M \setminus B(x_0, 1)$.

Hence, we may identify the manifold $(M, f_\varepsilon^2 g_\delta)$ to the manifold M_ε of Lemma 2.2 above obtained by glueing $M \setminus B(x_0, 1)$ to $\mathbb{S}^n \setminus D_\varepsilon$. Lemma 2.2 tells us that $\lambda_k(g_\varepsilon)$ converges, as $\varepsilon \rightarrow 0$, to $\lambda_k(h_\delta)$. It is also clear that the volume $V(g_\varepsilon)$ of (M, g_ε) converges to the volume of (\mathbb{S}^n, h_δ) . Therefore, there exists $\varepsilon > 0$ such that

$$\lambda_k(g_\varepsilon)V(g_\varepsilon)^{2/n} \geq \lambda_k(h_\delta)V(h_\delta)^{2/n} - \delta \geq (1+\delta)^{-2(n+2)}(\lambda_k^c(\mathbb{S}^n, [g_s]) - \delta) - \delta.$$

Now, using classical density results and the observation O2 above, we may find a smooth function \bar{f}_ε on M so that the smooth metric $\bar{g}_\varepsilon = \bar{f}_\varepsilon^2 g_\delta$ is $(1+\delta)$ -quasi-isometric to $g_\varepsilon = f_\varepsilon^2 g_\delta$. As g_δ is $(1+\delta)$ -quasi-isometric to g , the observation O1 above tells us that the metric $g'_\varepsilon = \bar{f}_\varepsilon^2 g$ is in fact $(1+\delta)^2$ -quasi-isometric to g_ε . Therefore, we have

$$\begin{aligned}\lambda_k(g'_\varepsilon)V(g'_\varepsilon)^{2/n} &\geq (1+\delta)^{-4(n+2)}\lambda_k(g_\varepsilon)V(g_\varepsilon)^{2/n} \\ &\geq (1+\delta)^{-6(n+2)}(\lambda_k^c(\mathbb{S}^n, [g_s]) - \delta) - \delta(1+\delta)^{-4(n+2)} \\ &= \lambda_k^c(\mathbb{S}^n, [g_s]) - O(\delta),\end{aligned}$$

with $O(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Since g'_ε is conformal to g , it follows, according the definition of $\lambda_k^c(M, [g])$,

$$\lambda_k^c(M, [g]) \geq \lambda_k^c(\mathbb{S}^n, [g_s]) - O(\delta).$$

As δ can be chosen arbitrarily small, we get the desired inequality:

$$\lambda_k^c(M, [g]) \geq \lambda_k^c(\mathbb{S}^n, [g_s]).$$

□

Proof of Theorem B: Let (M, g) be a n -dimensional compact Riemannian manifold of volume 1. Let k be a positive integer and ρ a positive real number. We endow \mathbb{S}^n with the metric

$$h_\rho = \frac{n\omega_n^{2/n}}{\lambda_k^c(M, [g]) + \rho} g_s,$$

so that its first positive eigenvalue becomes equal to $\lambda_k^c(M, [g]) + \rho$ (recall that $\lambda_1(g_s) = n\omega_n^{2/n}$), and consider a metric $g_\rho \in [g]$ on M such that

$$\lambda_k(g_\rho) \geq \lambda_k^c(M, [g]) - \rho/2.$$

Let δ be a sufficiently small positive real number so that

$$(1 + \delta)^{2(n+1)} \lambda_k^c(M, [g]) \leq (1 + \delta)^{-2(n+1)} (\lambda_k^c(M, [g]) + \rho).$$

We apply Lemma 2.3 to get a metric $g_{\rho, \delta}$ on M , a metric $h_{\rho, \delta} \in [g_s]$ on \mathbb{S}^n , and a constant $r_\delta > 0$ such that

- $g_{\rho, \delta}$ is $(1 + \delta)$ -quasi-isometric to g_ρ and $h_{\rho, \delta}$ is $(1 + \delta)$ -quasi-isometric to h_ρ ,
- $\forall \varepsilon \in (0, r_\delta)$, the geodesic balls $B(x_0, \varepsilon) \subset (M, g_{\rho, \delta})$ and $D(x_1, \varepsilon) \subset (\mathbb{S}^n, h_{\rho, \delta})$ are isometric to a Euclidean ball, where x_0 and x_1 are two given points of M and \mathbb{S}^n respectively.

As in the proof of Theorem A, we notice that, for any $\varepsilon \in (0, r_\delta)$, the closed ball $(\bar{B}(x_0, \varepsilon), g_{\rho, \delta})$ is conformally equivalent to $(\mathbb{S}^n \setminus D(x_1, \varepsilon), h_{\rho, \delta})$, and that there exists a piecewise smooth function f_ε on M which is equal to 1 on $M \setminus B(x_0, \varepsilon)$ and such that $(\bar{B}(x_0, \varepsilon), f_\varepsilon^2 g_{\rho, \delta})$ is isometric to $(\mathbb{S}^n \setminus D(x_1, \varepsilon), h_{\rho, \delta})$. Now, the manifold $(M, g_\varepsilon = f_\varepsilon^2 g_{\rho, \delta})$ is identified to the manifold M_ε of Lemma 2.1 obtained by glueing $(M \setminus B(x_0, \varepsilon), g_{\rho, \delta})$ to $(\mathbb{S}^n \setminus D(x_1, \varepsilon), h_{\rho, \delta})$. According to this lemma, the spectrum of (M, g_ε) converges, as ε goes to 0, to the reordered union of the spectra of $(M, g_{\rho, \delta})$ and $(\mathbb{S}^n, h_{\rho, \delta})$. From the construction, we have the following inequalities:

$$\lambda_k(g_{\rho, \delta}) \leq (1 + \delta)^{2(n+1)} \lambda_k(g_\rho) \leq (1 + \delta)^{2(n+1)} \lambda_k^c(M, [g]),$$

and

$$\lambda_1(h_{\rho, \delta}) \geq (1 + \delta)^{-2(n+1)} (\lambda_1(h_\rho) \geq (1 + \delta)^{-2(n+1)} (\lambda_k^c(M, [g]) + \rho).$$

Hence, from the smallness condition above satisfied by δ , we have

$$\lambda_k(g_{\rho, \delta}) \leq \lambda_1(h_{\rho, \delta}).$$

Consequently, the lowest $(k + 2)$ eigenvalues in $\text{Spec}(g_{\rho,\delta}) \cup \text{Spec}(h_{\rho,\delta})$ are:

$$\lambda_0(h_{\rho,\delta}), \lambda_0(g_{\rho,\delta}), \lambda_1(g_{\rho,\delta}), \lambda_2(g_{\rho,\delta}), \dots, \lambda_k(g_{\rho,\delta}).$$

Thus,

$$\lim_{\varepsilon \rightarrow 0} \lambda_{k+1}(g_\varepsilon) = \lambda_k(g_{\rho,\delta}).$$

On the other hand, the volume of (M, g_ε) converges, as $\varepsilon \rightarrow 0$, to $V(g_{\rho,\delta}) + V(h_{\rho,\delta})$. Since $g_{\rho,\delta}$ and $h_{\rho,\delta}$ are $(1 + \delta)$ -quasi-isometric to g_ρ and h_ρ respectively, we have

$$\lambda_k(g_{\rho,\delta}) \geq (1 + \delta)^{-2(n+1)} \lambda_k(g_\rho) \geq (1 + \delta)^{-2(n+1)} (\lambda_k^c(M, [g]) - \rho/2),$$

and

$$\begin{aligned} V(g_{\rho,\delta}) + V(h_{\rho,\delta}) &\geq (1 + \delta)^{-n} (V(g_\rho) + V(h_\rho)) \\ &= (1 + \delta)^{-n} (1 + \frac{n^{n/2} \omega_n}{(\lambda_k^c(M, [g]) + \rho)^{n/2}}). \end{aligned}$$

Therefore, there exists a positive ε such that

$$\begin{aligned} \lambda_{k+1}(g_\varepsilon)^{n/2} V(g_\varepsilon) &\geq (1 + \delta)^{-n(n+2)} \times \\ &\times (\lambda_k^c(M, [g]) - \rho)^{n/2} (1 + \frac{n^{n/2} \omega_n}{(\lambda_k^c(M, [g]) + \rho)^{n/2}}). \end{aligned}$$

As in the proof of Theorem A, we use the observations O1 and O2 to get a smooth metric \bar{g}_ε conformal to g and $(1 + \delta)^2$ -quasi-isometric to g_ε . Hence,

$$\lambda_{k+1}^c(M, [g])^{n/2} \geq \lambda_{k+1}(\bar{g}_\varepsilon)^{n/2} V(\bar{g}_\varepsilon) \geq (1 + \delta)^{-2n(n+2)} \lambda_{k+1}(g_\varepsilon)^{n/2} V(g_\varepsilon),$$

and then,

$$\begin{aligned} \lambda_{k+1}^c(M, [g])^{n/2} &\geq \\ &\geq (1 + \delta)^{-3n(n+2)} (\lambda_k^c(M, [g]) - \rho)^{n/2} (1 + \frac{n^{n/2} \omega_n}{(\lambda_k^c(M, [g]) + \rho)^{n/2}}), \end{aligned}$$

which gives, as $\delta \rightarrow 0$ and, then, $\rho \rightarrow 0$,

$$\lambda_{k+1}^c(M, [g])^{n/2} \geq \lambda_k^c(M, [g])^{n/2} + n^{n/2} \omega_n.$$

□

Proof of Theorem C: Let M_γ be a compact orientable surface of genus γ and let k be a natural integer. Given a positive real number δ , let g_δ be a Riemannian metric of area one on M_γ such that

$$\lambda_k(g_\delta) > \lambda_k^{top}(\gamma) - \delta/2.$$

Let us attach to M_γ a "thin" handle of radius $\varepsilon > 0$ and length $l > 0$ as described in [A]. After smoothing, we get a compact surface $M_{\gamma+1}$ of genus $(\gamma + 1)$ endowed with a Riemannian metric $g_{\delta,\varepsilon,l}$ such that, when ε goes to zero,

- the spectrum of the Laplacian of $g_{\delta,\varepsilon,l}$ converges to the reordered union of the spectrum of (M_γ, g_δ) and the spectrum of the segment $[0, l]$ for the Laplacian with Dirichlet boundary condition.
- the area $V(g_{\delta,\varepsilon,l})$ of $g_{\delta,\varepsilon,l}$ converges to 1,

Choosing l sufficiently small, one can suppose that the first Dirichlet eigenvalue of $[0, l]$ is greater than $\lambda_k^{top}(\gamma)$. Hence, the lowest $(k+1)$ eigenvalues in $Spec(M_\gamma, g_\delta) \cup Spec([0, l])$ are:

$$\lambda_0(g_\delta), \lambda_1(g_\delta), \lambda_2(g_\delta), \dots, \lambda_k(g_\delta),$$

and then

$$\lim_{\varepsilon \rightarrow 0} \lambda_k(g_{\delta,\varepsilon,l}) = \lambda_k(g_\delta).$$

Therefore, there exist two positive constants ε and l such that

$$\lambda_k(g_{\delta,\varepsilon,l}) \geq \lambda_k(g_\delta) - \delta/2 > \lambda_k^{top}(\gamma) - \delta$$

and

$$V(g_{\delta,\varepsilon,l}) > 1 - \delta.$$

Consequently

$$\lambda_k^{top}(\gamma + 1) \geq \lambda_k(g_{\delta,\varepsilon,l}) V(g_{\delta,\varepsilon,l}) \geq (\lambda_k^{top}(\gamma) - \delta)(1 - \delta).$$

In conclusion, we have

$$\lambda_k^{top}(\gamma + 1) \geq \lambda_k^{top}(\gamma).$$

□

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